State Dependence Can Explain the Risk Aversion Puzzle

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Risk aversion functions extracted from observed stock and option prices can be negative, as shown by Aït-Sahalia and Lo (2000), *Journal of Econometrics* 94: 9–51; and Jackwerth (2000), *The Review of Financial Studies* 13(2), 433–51. We rationalize this puzzle by a lack of conditioning on latent state variables. Once properly conditioned, risk aversion functions and pricing kernels are consistent with economic theory. To differentiate between the various theoretical explanations in terms of heterogeneity of beliefs or preferences, market sentiment, state-dependent utility, or regimes in fundamentals, we calibrate several consumption-based asset pricing models to match the empirical pricing kernel and risk aversion functions at different dates and over several years. (*JEL* G12, G13)

1. Introduction

Several researchers have extracted risk aversion functions or preference parameters from observed stock and bond prices. In a complete market economy, which implies the existence of a representative investor, absolute-risk aversion can be evaluated for any state of wealth in terms of the historical and risk-neutral distributions. Aït-Sahalia and Lo (2000) and Jackwerth (2000) have proposed nonparametric approaches to recover risk aversion functions across wealth states from observed stock and option prices. Rosenberg and Engle (2002); Garcia, Luger, and Renault (2003) and Bliss and Panigirtzoglou (2004) have estimated preference parameters basedon parametric asset pricing models with several specifications of the utility function.

These efforts to exploit prices of financial assets to recover fundamental economic parameters have produced puzzling results. Aït-Sahalia and Lo (2000) find that the nonparametrically implied function of relative risk aversion varies significantly across the range of S&P 500 index values, from 1 to 60, and is U-shaped. Jackwerth (2000) finds also that the implied absolute-risk aversion function is U-shaped around the current forward price, but even then it can become negative. Parametric empirical estimates of the coefficient of relative risk aversion also show considerable variation. Rosenberg and Engle (2002) report values ranging from 2.36 to 12.55 for a power utility pricing kernel across time,[1] while Bliss and Panigirtzoglou (2004) estimate average values between 2.33 and 11.14 for the same S&P 500 index for several option maturities. Garcia, Luger, and Renault (2003) estimate a consumption-based asset pricing model with regime-switching fundamentals and Epstein and Zin (1989) preferences. The estimated parameters for risk aversion and intertemporal substitution are reasonable, with average values of 0.6838 and 0.8532, respectively, over the 1991–1995 period.[2]

The main goal of this paper is to reconcile these facts with economic theory. We provide a unifying explanation for the various reported results. While several statistical, behavioral, and theoretical explanations have been proposed, they lack a common thread to understand the fundamental reason why these results are obtained. We argue that state dependence lies at the heart of the nonparametric risk aversion puzzles or the variability of the parametric estimates. State dependence refers to a conditioning set of state variables, typically unobservable to either the researcher or the investors or both. By assuming joint lognormality for the returns on a stock index and a general stochastic discount factor, conditional to this set of state variables, we derive an option pricing formula that admits as special cases several option pricing formulas based on equilibrium or absence of arbitrage. To obtain the risk-neutral distribution, we appeal to the Breeden and Litzenberger (1978) formula relating the second derivative of the option pricing formula to the risk-neutral distribution.

The conditioning set differs whether we consider the future state variables or the past state variables. We argue that these conditioning issues rationalize the risk aversion puzzles in the sense that, once properly conditioned, the pricing kernel and risk aversion functions are consistent with economic theory. Without the conditioning information, the objective distribution acquires fatter tails, while the risk-neutral distribution becomes more negatively skewed. As a result, a risk aversion puzzle is inferred.

Two main theoretical explanations have been put forth to explain the risk aversion puzzle. Ziegler (2007) argues that heterogeneity of beliefs is the most

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[1] The authors also estimate an orthogonal polynomial pricing kernel and find that it exhibits some of the risk-aversion characteristics noted by Jackwerth (2000), with a region of negative absolute-risk aversion over the range from 4 to 2% for returns and an increasing absolute-risk aversion for returns greater than 4%.

[2] The authors also estimate a CRRA expected utility model and find a similar variability of the estimates as in the related studies. The average value is 7.2 over the 1991–1995 period with a standard deviation of 4.83.

likely cause of the U-shaped pattern. He writes a model with two types of CARA agents with weights ϕ and $1 - \phi$ in the economy. Both types view future asset prices as lognormally distributed, but differ in their estimates of the mean and variance of the process. The estimates suggest that there are two regimes, but the interpretation that beliefs are heterogeneous may not be warranted.[3] The mixture could be interpreted instead as two regimes in the fundamentals and the parameter ϕ as the probability of being in one regime. Brown and Jackwerth (2000)[4] assume instead that the representative investor has state-dependent utility, which they interpret as a generalized utility (external habit formation as in Campbell and Cochrane, 1999) or simply a utility with volatility as an additional state variable. They retain the latter to construct a pricing kernel that is a weighted average of two pricing kernels associated with high and low volatility. Again, this high and low volatility could be associated with two volatility regimes for the endowment process in the economy.

Behavioral explanations have also been put forward, mainly by Shefrin (2000, 2005), but amount to a model with heterogeneous beliefs similar to Ziegler (2007) with two types of investors. More convincingly, Han (2004) establishes an empirical link between the risk-neutral skewness of index return and investor sentiment measured by an institutional investor index. Since the index risk-neutral skewness is related to the slope of the pricing kernel with respect to the index return, market sentiment seems to affect the pricing kernel.

Another contribution of the paper is to propose two consumption-based asset pricing models that include state variables that can help sort out the various hypotheses about the source of the puzzle. The models will permit determination of whether state dependence manifests itself in preferences, economic fundamentals, or beliefs. The first model generalizes two existing models based on recursive utility. Garcia, Luger, and Renault (2001) proposed a general asset pricing model with Epstein and Zin (1989) preferences where the joint process of consumption and dividends is lognormal, conditional to a set of state variables.[5] In Melino and Yang (2003), investors' preferences are also based on the Epstein and Zin model, but they are state-dependent. We derive an option pricing formula in closed form for this generalized Epstein-Zin asset pricing model and characterize analytically the risk-neutral and objective distributions based on several assumptions about the conditioning information. We also derive the two distributions and a corresponding option pricing formula in a model with belief-dependent utility, with external habit formation proposed by Veronesi (2004), where the uncertainty is again characterized by a set of state variables.

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[3] Liu et al. (2004) propose a parametric closed-form transformation from risk-neutral to real-world distributions based on a mixture of two lognormal densities, but without any particular economic interpretation.

[4] We refer to the date of the first issue of this working paper, but some material is found only in the updated version of 2004.

[5] David and Veronesi (2002) also propose an incomplete information model where investors' uncertainty about the dividend process explains the intertemporal variation in the slope and curvature of implied volatility curves.

To illustrate the risk aversion puzzles, we assume that these state variables follow a discrete-state Markov process with two states. In these models, by construction, the risk aversion functions are consistent with economic theory within each regime, since the regime is observed by investors. However, as in Jackwerth (2000), we obtain negative estimates of the risk aversion function in some states of wealth. The pricing kernel function across wealth states also exhibits a puzzle even though this function is decreasing within each regime. We also calibrate the models to S&P 500 futures and options data over the 1988 to 1995 period. We find that in any given day, evidence can alternate in support of one particular model. For example, the recursive utility model with changing risk aversion seems to fit better the data just after the 1987 stock market crash. However, in average over a year, the external habit model with state-dependent beliefs seems to consistently better approximate the empirical pricing kernel and risk aversion functions.

In the discussion so far, we have left all estimation issues aside. Indeed, the puzzle comes from the fact that the states are unobservable at least by the researcher if not by investors. Several statistical methodologies are possible to recover the objective distribution of future returns (on the underlying index), given current ones. As emphasized by Jackwerth (2000), the choice of a particular estimation strategy should not have any impact on the documented puzzles. For instance, a kernel estimation will be valid under very general stationarity and mixing conditions. While historical probabilities p_j are recovered from a time series of underlying index returns, risk-neutral probabilities p_j^* will be backed out in cross sections from a set of observed option prices written on the same index. In Jackwerth (2000), the risk-neutral distribution is recovered from prices on European call options written on the S&P 500 index by applying a variation of the nonparametric method introduced in Jackwerth and Rubinstein (1996, 2004). The basic idea of this method is to search for the smoothest risk-neutral distribution according to a specific objective function, which at the same time explains the option prices. However, Jackwerth and Rubinstein observe that the implied distributions are rather independent of the choice of the objective function when a sufficiently high number of options is available.[6]

Finally, Brown and Jackwerth (2000) put aside other possible market explanations for the puzzles, such as lack of liquidity of out-of-the-money options, market frictions due to margin account requirements, or the difficulty of hedging when selling out-of-the-money puts.

The remainder of this paper is organized as follows. In Section 2, we analyze the pricing kernel and risk aversion puzzles. In Section 3, we present a general asset pricing model with state variables and derive a formula for the price of a European call option on a stock index. We further characterize the objective and risk-neutral probabilities across wealth states in this model. Section 4

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[6] They notice that "as few as 8 option prices seem to contain enough information to determine the general shape of the implied distribution" and that "at the extreme, the constraints themselves will completely determine the solution."

introduces two economic models, where economic fundamentals, preferences, and investors' beliefs can be state-dependent. Section 5 contains the empirical illustrations of the puzzles based on a calibration of these economic models. In Section 6, we provide a discussion to relate our framework to other approaches found in the literature. Section 7 concludes.

2. The Pricing Kernel and Risk Aversion Puzzles

In this section, we recall the puzzles put forward by Aït-Sahalia and Lo (2000) and Jackwerth (2000).

2.1 Theoretical underpinnings

Under very general nonarbitrage conditions (Hansen and Richard, 1987), the time t price of an asset that delivers a payoff g_{t+1} at time $(t+1)$ is given by:

$$p_t = E_t [m_{t+1} g_{t+1}], \qquad (1)$$

where $E_t [.]$ denotes the conditional expectation operator given investors' information at time t. Any random variable m_{t+1} conformable to Equation (1) is called an admissible stochastic discount factor (SDF). Among the admissible SDFs, only one denoted by m_{t+1}^* is a function of available payoffs. It is the orthogonal projection of any admissible SDF on the set of payoffs. If some rational investor is able to separate its utility over current and future values of consumption,

$$U [C_t, C_{t+1}] = u(C_t) + \beta u(C_{t+1}), \qquad (2)$$

the first-order condition for an optimal consumption and portfolio choice will imply that m_{t+1}^* coincides with the projection of $\beta \frac{u'(C_{t+1})}{u'(C_t)}$ on the set of payoffs. Therefore, through a convenient aggregation argument, concavity of utility functions should imply that m_{t+1}^* is decreasing in current wealth.

Moreover, as shown by Hansen and Richard (1987), no arbitrage implies almost sure positivity of m_{t+1}^*. Therefore, $m_{t+1}^*/E_t m_{t+1}^*$ can be interpreted as the density function of the risk-neutral probability distribution with respect to the historical one. In case of a representative investor with preferences conformable to Equation (2), we deduce:

$$\frac{m_{t+1}^*}{E_t m_{t+1}^*} = \frac{u'(C_{t+1})}{E_t u'(C_{t+1})}.$$

Therefore,

$$\frac{\partial Log m_{t+1}^*}{\partial C_{t+1}} = \frac{u''(C_{t+1})}{u'(C_{t+1})} \qquad (3)$$

is the opposite of the Arrow-Pratt index of absolute-risk aversion (ARA) of the investor.

2.2 The puzzles

For sake of simplicity, it is convenient to analyze these puzzles in a finite state space framework. If $j = 1, \ldots, n$ denote the possible states of nature, we get the density function of the risk-neutral distribution probability with respect to the historical one as:

$$\frac{m_{t+1}^*}{E_t m_{t+1}^*} = \frac{p_j^*}{p_j} \text{ in state } j, \tag{4}$$

where p_j^* is the risk-neutral probability across wealth states $j = 1, \ldots, n$ and p_j is the corresponding historical probability. Brown and Jackwerth (2000) use the formula in Equation (4) to empirically derive the pricing kernel function from realized returns on the S&P 500 index and option prices on the index over a post-1987 period. For the center wealth states (over the range of 0.97 to 1.03 with wealth normalized to one), they found a pricing kernel function that is increasing in wealth. This is the so-called pricing kernel puzzle.

As explained in Section 2.1, the increasing nature of the pricing kernel function in wealth is puzzling because it is akin to a convex utility function for a representative investor, which is obviously inconsistent with the general assumption of risk aversion. From Equation (3), the ARA coefficient can actually be computed through a log-derivative of the pricing kernel. By using Equation (4), we deduce

$$ARA = -\frac{u''(C_{t+1})}{u'(C_{t+1})} = \frac{p_j'}{p_j} - \frac{p_j^{*'}}{p_j^*}, \tag{5}$$

where p_j' and $p_j^{*'}$ are of the derivatives of p_j and p_j^* with respect to aggregate wealth in state j.

Jackwerth (2000) observes that the absolute-risk aversion functions computed from Equation (5) dramatically change shape around the 1987 stock market crash. Prior to the crash, they are positive and decreasing in wealth, which is consistent with standard assumptions made in economic theory about investors' preferences. After the crash, they are partially negative and increasing (see Figure 3 in Jackwerth, 2000). This result is called the risk aversion puzzle. One component of it is equivalent to the pricing kernel puzzle: ARA should be positive as the pricing kernel should be decreasing in aggregate wealth. Additionally, even when there is no pricing kernel puzzle (positive ARA), there remains a risk aversion puzzle when ARA is increasing in wealth. While the pricing kernel puzzle is only observed for the center wealth states, the risk aversion puzzle (increasing ARA) remains for larger levels of wealth. Without any discretization of wealth states, Aït-Sahalia and Lo (2000) documented similar empirical puzzles for implied risk aversion.

3. A General Asset Pricing Model with State Variables

In this section, we specify a general dynamic asset pricing model with latent variables. This model will allow us to extract analytically the risk-neutral and historical probabilities (p_j^* and p_j, respectively) across wealth states. Therefore, we will be able to deduce the absolute-risk aversion function as well as the pricing kernel function. To obtain the risk-neutral distribution, we will appeal to the Breeden and Litzenberger (1978) formula relating the second derivative of the option pricing formula to the risk-neutral distribution. Therefore, we first develop an option pricing formula in this general dynamic asset pricing framework and then provide expressions for the historical and risk-neutral distributions.

In order to specify a dynamic asset pricing model in discrete time, our focus of interest will be the dynamic properties of a positive stochastic discount factor (SDF) denoted by $m_{t,T}$. Since agents typically observe more than the econometrician, the information set I_t at time t may contain not only past values of prices and payoffs, but also some latent state variables.

Extending the Hansen and Richard (1987) setting to an intertemporal framework and applying the law of iterated expectations, the log-SDFs necessarily fulfill

$$\log m_{t,T_2} = \log m_{t,T_1} + \log m_{T_1,T_2} \text{ , for } t < T_1 < T_2, \tag{6}$$

and therefore, $m_{t,T} = \prod_{\tau=t}^{T-1} m_\tau$, with: $m_\tau = m_{\tau-1,\tau}$. To show the generality of the latent state approach, we directly specify the time-series properties of the stochastic process m_t, $t = 1, 2, \ldots, T$.

3.1 An option pricing formula with state variables

The key feature of this asset pricing model is an assumption about the sequence $(m_\tau)_{1 \leq \tau \leq T}$ of unit period SDFs, which amounts to a factor structure in the longitudinal dimension. A number of state variables summarize the stochastic dependence of the consecutive SDFs, in the sense that, given the state variables, they are mutually conditionally independent. The same assumption is made about the sequence of consecutive returns of the primitive asset of interest on which options are written. Therefore, in terms of the joint distribution of m_t and returns on a given asset price S_t, we maintain the following assumption.[7]

Assumption 1. *The variables* $(m_{\tau+1}, \frac{S_{\tau+1}}{S_\tau})_{1 \leq \tau \leq T-1}$ *are conditionally serially independent given the path* $U_1^T = (U_t)_{1 \leq t \leq T}$ *of a vector* U_t *of state variables.*

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[7] This model extends the bond pricing model of Constantinides (1992). In the latter, Assumption 1 is only maintained for the SDF sequence (m_t). Resulting bond prices were, therefore, deterministic functions of the state variables, and Assumption 1 becomes trivial with S_t viewed as a bond price. A second extension relates to the processes considered for the state variables. While Constantinides (1992) considers only AR(1) processes, our setting accommodates any process. In particular, we have in mind Markov switching regime models, which can capture any kind of stochastic volatility and jumps in the return process as well as in the volatility.

Assumption 1 expresses that the dynamics of the returns are driven by the state variables. A similar assumption is made in common stochastic volatility models (the stochastic volatility process being the state variable) when standardized returns are assumed to be independent.

Assumption 2. *The process* $\left(m_t, \frac{S_t}{S_{t-1}}\right)$ *does not Granger-cause the state variables process* (U_t).

This assumption states that the state variables are exogenous. For common stochastic volatility or hidden Markov processes, such an exogeneity assumption is usually maintained to make the standard filtering strategies valid. It should be noted that this exogeneity assumption does not preclude instantaneous causality relationships, such as a leverage effect.

Assumption 3. *The conditional probability distribution of* $(\log m_{t+1}, \log \frac{S_{t+1}}{S_t})$ *given* U_1^{t+1} *is a bivariate normal;*

$$
\begin{bmatrix} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{bmatrix} \Big| U_1^{t+1} \rightsquigarrow \mathcal{N} \left[\begin{pmatrix} \mu_{mt} \\ \mu_{st} \end{pmatrix}, \begin{pmatrix} \sigma_{mt}^2 & \sigma_{mst} \\ \sigma_{mst} & \sigma_{st}^2 \end{pmatrix} \right].
$$

Assumption 3 is a very general version of the mixture of normals model. A maintained assumption will be that investors observe U_t at time t, so that the conditioning information in the expectation operator (Proposition 3.1) is

$$
I_t = \sigma \left[m_\tau, S_\tau, U_\tau, \tau \le t \right]. \tag{7}
$$

Proposition 3.1. *Under Assumptions 1, 2, and 3, the price of a European call option* π_t *is given by*

$$
\frac{\pi_t}{S_t} = \pi_t(x_t) = E_t \left\{ Q_{ms}(t, T) \, \Phi\left(d_1(x_t)\right) - \frac{\tilde{B}(t, T)}{B(t, T)} e^{-x_t} \Phi\left(d_2(x_t)\right) \right\},
$$

where $x_t = \log \frac{S_t}{K B(t,T)}$, $B(t, T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T, and*

$$
d_1(x) = \frac{x}{\bar{\sigma}_{t,T}} + \frac{\bar{\sigma}_{t,T}}{2} + \frac{1}{\bar{\sigma}_{t,T}} \log \left[Q_{ms}(t, T) \frac{B(t, T)}{\tilde{B}(t, T)} \right],
$$

$$
d_2(x) = d_1(x) - \bar{\sigma}_{t,T}, \quad \bar{\sigma}_{t,T}^2 = \sum_{\tau=t}^{T-1} \sigma_{s\tau}^2,
$$

and

$$
\tilde{B}(t, T) = \exp\left(\sum_{\tau=t}^{T-1} \mu_{m\tau+1} + \frac{1}{2} \sum_{\tau=t}^{T-1} \sigma_{m\tau}^2 \right),
$$

$$Q_{ms}(t, T) = \tilde{B}(t, T) \exp\left(\sum_{\tau=t}^{T-1} \sigma_{ms\tau+1}\right) E\left[\frac{S_T}{S_t} \middle| U_1^T\right].$$

Proof. See Appendix. ∎

This formula admits as special cases several option pricing formulas based on equilibrium or absence of arbitrage.[8] Regarding the equilibrium approach, our setting is very general since it is based on a stochastic model for the SDF, which does not rely on restrictive assumptions about preferences, endowments, or agent heterogeneity. Moreover, the factorization for the SDF is more general than the usual product of intertemporal marginal rates of substitution in time-separable utility models. Indeed, as we will see in the next section, the SDF allows accommodation of nonseparable or state-dependent preferences or beliefs.

3.2 Characterizing the objective and risk-neutral probabilities

The option pricing formula derived in the previous section allows us to recover the risk-neutral probability distribution by appealing to Breeden and Litzenberger (1978). Assumptions 1 to 3 provide us also with the elements to derive the objective probability distributions, both conditional and unconditional.

3.2.1 Objective probabilities across wealth states. Under Assumption 3, given the trajectory of the state variables, the log stock return, $\log R_{t,T} = \frac{S_T + D_T}{S_t}$, is normally distributed:

$$\log R_{t,T} | U_1^T \rightsquigarrow N\left(\mu_r, \sigma_r^2\right). \tag{8}$$

To derive the objective probabilities across wealth states, we have to discretize the objective density of $R_{t,T}$ onto wealth states. We denote:

$$p_j\left(U_1^T\right) = P\left(R_{t,T} = r_j | U_1^T\right), \ j = 1, \ldots, n, \tag{9}$$

the objective probabilities across wealth states conditionally to the trajectory of the state variables. This is to be contrasted with the objective probabilities that do not condition on the trajectory of the state variables:

$$p_j = E\left\{P\left(R_{t,T} = r_j | U_1^T\right)\right\}, \ j = 1, \ldots, n. \tag{10}$$

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[8] It is a conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate $\bar{r}_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau+1)$ and the cumulated volatility $\bar{\sigma}_{t,T}$. It nests three well-known models: the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic; the Hull and White (1987) stochastic volatility extension, since $\bar{\sigma}_{t,T}^2 = Var\left[\log \frac{S_T}{S_t} | U_1^T\right]$ corresponds to the integrated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting; the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992).

We can also compute the objective probabilities across wealth states given U_1^t, the past of the state variables:

$$p_j \left(U_1^t \right) = E_t p_j \left(U_1^T \right), \, j = 1, \ldots, n, \tag{11}$$

where the expectation is taken with respect to U_{t+1}^T. Once these probabilities are computed, it is straightforward to deduce the marginal probabilities:

$$p_j = E p_j \left(U_1^t \right), \, j = 1, \ldots, n, \tag{12}$$

where the expectation is taken with respect to U_t. These two sets of probabilities differ by the conditioning information taken into account. The first set conditions on the whole trajectory of state variables, past and future. This is to compare our framework to other papers, such as Brown and Jackwerth (2000); and Ziegler (2007), where they simulate future trajectories of the processes and estimate the distributions by kernel. In our setting, we are able to characterize the distributions analytically.

The second set of probabilities in Equations (11) and (12) is meant to condition on the available information at time t and to capture the potential informational gap between the econometrician or the researcher and the investor. While the latter may observe the past states U_1^t in t, the former will not have knowledge of it and will compute an unconditional probability.

3.2.2 Risk-neutral probabilities.
In a complete market, Ross (1976) shows that one can recover the risk-neutral distribution from a set of European option prices. This distribution turns out to be unique. Breeden and Litzenberger (1978) give an exact formula for the risk-neutral distribution:[9]

$$\left(\frac{\partial^2 \pi_t}{\partial^2 K} \right)_{K=x} = e^{-r_f(T-t)} f^* (x), \tag{13}$$

where r_f is the risk-free rate, and $f^* (x)$ is the risk-neutral density of S_T at x. The option pricing formula $\pi_t = E_t C_{op} \left(U_1^T \right)$ will be of the form specified in Proposition 3.1. Therefore, applying the Breeden and Litzenberger (1978) formula gives

$$\left(\frac{\partial^2 \pi_t}{\partial^2 K} \right)_{K=x} = E_t \left(\frac{\partial^2 C_{op} \left(U_1^T \right)}{\partial^2 K} \right)_{K=x}. \tag{14}$$

We denote

$$\frac{\partial^2 C_{op} \left(U_1^T \right)}{\partial^2 K} = \widetilde{B} (t, T) f^* \left(x | U_1^T \right), \tag{15}$$

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[9] Ait-Sahalia and Lo (1998) provide a nonparametric method to estimate the State-Price Density.

where $f^*\left(x|U_1^T\right)$ represents the risk-neutral density of S_T given the trajectory of state variables. As a result, the risk-neutral density of $R_{t,T} = \frac{S_T + D_T}{S_t}$ conditional on the trajectory of state variables can be obtained using a simple transformation of $f^*\left(.|U_1^T\right)$. To derive the risk-neutral probabilities across wealth states, we have to discretize the risk-neutral probability distribution of $R_{t,T} = \frac{S_T + D_T}{S_t}$ onto the same wealth space that we employed for the objective distribution. If p^* represents the risk-neutral probability distribution of $R_{t,T} = \frac{S_T + D_T}{S_t}$ conditional on the trajectory of state variables, we denote

$$p_j^*\left(U_1^T\right) = P^*\left(R_{t,T} = r_j|U_1^T\right), \ j = 1, \ldots, n, \tag{16}$$

the risk-neutral probabilities across wealth states given the whole trajectory of the state variables. Similar to the objective probabilities, we can compute

$$p_j^* = E\{P^*\left(R_{t,T} = r_j|U_1^T\right)\}, \ j = 1, \ldots, n, \tag{17}$$

which do not condition on the trajectory of the state variables.

We also compute the risk-neutral probabilities given the past of the state variables U_1^t:

$$p_j^*\left(U_1^t\right) = E_t p_j^*\left(U_1^T\right). \tag{18}$$

The marginal risk-neutral probabilities across wealth states are computed by taking the expectation of Equation (18) as follows:

$$p_j^* = E p_j^*\left(U_1^t\right), \ j = 1, \ldots, n, \tag{19}$$

where the expectation is taken with respect to U_1^t. Again, the probabilities in Equations (18) and (19) characterize the potential information gap between the investor and the researcher, respectively.

Our main objective is to show that such a state-variable framework will be able to reproduce the risk aversion and pricing kernel puzzles. However, to provide an economic content to the source of the puzzles, we need to specify equilibrium models of the SDF that are interpretable in terms of economic fundamentals, preferences, or beliefs.

4. Economies with Regime Shifts

In the previous section, we argued that the risk aversion puzzle could be rationalized by a general dependence of the stochastic discount factor on state variables. Some authors have proposed economic explanations for the risk aversion puzzle. For example, Shefrin (2000, 2005); and Ziegler (2007) make a case for the heterogeneity of beliefs, while Brown and Jackwerth (2000) rationalize the puzzle through state-dependent utility.[10] However, ultimately, the puzzles

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[10] Recently, Bakshi and Madan (2006) appeal to a heterogeneous population of investors, some taking long positions, others short positions, in the equity market-index to explain the same puzzle.

are often illustrated by using a mixture of normals, which makes it difficult to identify the actual source of the puzzles. In this section, we propose two main classes of equilibrium models with dependence on state variables, one based on recursive utility, another on external habit formation. In these models, preferences, fundamentals, or beliefs may change with the state variables, allowing for the potential identification of the actual source among several alternatives.

4.1 State-dependent preferences and fundamentals in a recursive utility framework

The recursive utility framework of Epstein and Zin (1989) leads them to the following SDF:

$$m_{t+1} = \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \right]^{\gamma} \left[\frac{1}{R_{mt+1}} \right]^{1-\gamma}, \tag{20}$$

where $\rho = 1 - \frac{1}{\sigma}$, σ is the elasticity of intertemporal substitution, and $\gamma = \frac{\alpha}{\rho}$ with $(1 - \alpha)$ is the index of relative risk aversion. With a two-state mixing variable U_{t+1}, $\log m_{t+1}$ appears as a mixture of two normal distributions in two cases. Garcia, Luger, and Renault (2001, 2003); and Melino and Yang (2003) considered recursive utility models with state dependence. In the former, there are regime shifts in fundamentals and the joint probability distribution of $\left(\log \frac{C_{t+1}}{C_t}, \log R_{mt+1} \right)$ is a mixture of normals, while in the latter, preference parameters are functions of U_{t+1}.[11]

Let us first assume, as in Melino and Yang (2003), that the three preference parameters β, α, and ρ are all state-dependent and then denoted as $\beta(U_t)$, $\alpha(U_t)$, and $\rho(U_t)$. While these values, known by the investor at time t, define her time t utility level, she does not know at this date the next coming values $\beta(U_{t+1})$, $\alpha(U_{t+1})$, and $\rho(U_{t+1})$. Therefore, the resulting SDF will be more complicated than just replacing α, β, and ρ in Equation (20) by their state-dependent value. Melino and Yang show that the SDF is

$$m_{t+1} = \left[\beta(U_t) \left(\frac{C_{t+1}}{C_t} \right)^{\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}} \right]^{\gamma(U_t)} R_{mt+1}^{\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1} P_t^{\frac{\alpha(U_t)}{\rho(U_{t+1})} - \frac{\alpha(U_t)}{\rho(U_t)}}, \tag{21}$$

where $\gamma(U_t) = \frac{\alpha(U_t)}{\rho(U_t)}$ and P_t is the time t price of the market portfolio. When $\beta(U_t)$, $\alpha(U_t)$, $\rho(U_t) = \rho(U_{t+1})$ are constants, this pricing kernel reduces to the Epstein and Zin SDF in Equation (20). By definition,

$$R_{mt+1} = \frac{P_{t+1} + C_{t+1}}{P_t}, \tag{22}$$

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[11] See also Gordon and St-Amour (2000) for an alternative way to introduce state dependence in preferences in a CCAPM framework.

while the underlying asset return is $\frac{S_{t+1}+D_{t+1}}{S_t}$. Asset prices P_t and S_t are then determined as discounted values of future payoffs by iterating on the following pricing formulas:

$$P_t = E_t\left[m_{t+1}\left(P_{t+1} + C_{t+1}\right)\right] \text{ and } S_t = E_t\left[m_{t+1}\left(S_{t+1} + D_{t+1}\right)\right]. \quad (23)$$

It can be shown that Assumptions 1 and 2 are implied by similar assumptions stated for the joint process $\left(\frac{C_{t+1}}{C_t}, \frac{D_{t+1}}{D_t}\right)$. Assumption 3 will then also be implied by a similar assumption about fundamentals.

Assumption 3′. *The conditional probability distribution of* $\left(\log\frac{C_{t+1}}{C_t}, \log\frac{D_{+1}}{D_t}\right)$ *given* U_1^{t+1} *is a bivariate normal:*

$$\begin{bmatrix} \log\frac{C_{t+1}}{C_t} \\ \log\frac{D_{+1}}{D_t} \end{bmatrix} \Bigg| U_1^{t+1} \rightsquigarrow N\left[\begin{pmatrix} \mu_{X_{t+1}} \\ \mu_{Y_{t+1}} \end{pmatrix}, \begin{pmatrix} \sigma_{X_{t+1}}^2 & \sigma_{XY,t+1} \\ \sigma_{XY,t+1} & \sigma_{Y_{t+1}}^2 \end{pmatrix}\right].$$

Proposition 4.1 below provides a European call option pricing formula for an asset pricing model that nests the models of Melino and Yang (2003); and Garcia, Luger, and Renault (2003). First, it is worth noticing that the equilibrium model characterizes the asset prices P_t and S_t as

$$\frac{P_t}{C_t} = \lambda\left(U_1^t\right) = E_t\left[m_{t+1}\left(1 + \lambda\left(U_1^{t+1}\right)\right)\frac{C_{t+1}}{C_t}\right],$$

$$\frac{S_t}{D_t} = \varphi\left(U_1^t\right) = E_t\left[m_{t+1}\left(1 + \varphi\left(U_1^{t+1}\right)\right)\frac{D_{t+1}}{D_t}\right].$$

Proposition 4.1. *Under Assumptions 1, 2, and 3′, the price of a European call option on a dividend-paying stock is given by*

$$\pi_t = E_t[S_t Q_{XY}(t, T)\Phi(d_1(x_t)) - K\widetilde{B}(t, T)\Phi(d_2(x_t))],$$

where $x_t = \log\frac{S_t}{KB(t,T)}$, $B(t, T) = E_t\left(\prod_{\tau=t}^{T-1} m_{\tau+1}\right)$ *is the time t price of a bond maturing at time T, and*

$$d_2(x_t) = \frac{x_t + \log\left(Q_{XY}(t, T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}{\sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}},$$

$$d_1(x_t) = d_2(x_t) + \sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}$$

with

$$Q_{XY}(t, T) = \widetilde{B}(t, T) E_t\left[\frac{(S_T + D_T)}{S_t}\bigg| U_1^T\right]\exp[\psi],$$

$$\widetilde{B}(t, T) = \exp\left[\sum_{\tau=t}^{T-1} \mu_{m\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1} \sigma_{m\tau}^2\right],$$

and

$$E_t\left[\frac{(S_T + D_T)}{S_t}\middle| U_1^T\right] = \exp\left[\log\frac{(\varphi(U_1^T) + 1)}{\varphi(U_1^{T-1})} + \sum_{\tau=t}^{T-2}\log\frac{\varphi(U_1^{\tau+1})}{\varphi(U_1^{\tau})}\right.$$
$$\left. + \sum_{\tau=t}^{T-1}\mu_{Y\tau+1} + \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2\right],$$

where

$$\psi = \sum_{\tau=t}^{T-1} a_{1\tau+1}\sigma_{XY,\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\sigma_{XY,\iota+1}\right),$$

$$\sum_{\tau=t}^{T-1}\mu_{m\tau+1} = \sum_{\tau=t}^{T-1}\gamma(U_\tau)\log(\beta(U_\tau)) + \left[\sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau)\right)\right]\log(C_t)$$
$$+ \sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - 1\right)\log\left(\frac{\lambda(U_1^{\tau+1}) + 1}{\lambda(U_1^\tau)}\right)$$
$$+ \sum_{\tau=t}^{T-1}\left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau)\right)\log\left(\lambda(U_1^\tau)\right)$$
$$+ \sum_{\tau=t}^{T-1} a_{1\tau+1}\mu_{X\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}\left(\sum_{\iota=t}^{\tau-1}\mu_{X\iota+1}\right),$$

$$\sum_{\tau=t}^{T-1}\sigma_{m\tau}^2 = \sum_{\tau=t}^{T-1} a_{1\tau+1}^2\sigma_{X\tau+1}^2 + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}^2\sum_{\iota=t}^{\tau-1}\sigma_{X\iota+1}^2$$
$$+ 2\sum_{t\leq i < j \leq T-1} a_{0i+1}a_{0j+1}\sum_{\iota=t}^{i-1}\sigma_{X\iota+1}^2.$$

with

$$a_{0\tau+1} = \frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - \gamma(U_\tau),$$

$$a_{1\tau+1} = \gamma(U_\tau)\left(\rho(U_\tau) - \frac{\rho(U_\tau)}{\rho(U_{\tau+1})}\right) + \left(\frac{\alpha(U_\tau)}{\rho(U_{\tau+1})} - 1\right).$$

Proof. See Appendix. ∎

If the preference parameters α, β, and ρ are constants, Proposition 4.1 collapses to the Garcia, Luger, and Renault (2003) option pricing formula. Note

that the definition of $\lambda \left(U_1^{t+1} \right)$ and $\varphi \left(U_1^{t+1} \right)$ is equivalent to $E_t Q_{XY}(t, t+1) = 1$, and $E_t \widetilde{B}(t, t+1) = B(t, t+1)$. This option pricing formula will help us compute the corresponding objective and risk-neutral probabilities as defined in the previous section.

4.2 An external habit model with state dependence in beliefs

Veronesi (2004) introduces the concept of belief-dependent utility functions. Consider an economy where $\mathcal{C}$ is a set of prizes and ϑ a set of states. Define a lottery $f: \vartheta \mapsto \Delta(\mathcal{C})$ assigning a probability distribution $\Delta(\mathcal{C})$ on $\mathcal{C}$ to each state $\upsilon \in \vartheta$. From Theorem 1 in Veronesi (2004), there exists a state-dependent utility function $u : \mathcal{C} \times \vartheta \mapsto \mathbb{R}$ and a subjective probability function $\pi(.)$ such that

$$U(C_t, \upsilon_t) = \sum_{\upsilon^i \in \vartheta} \pi_t(\upsilon^i) u(C_t, t | \upsilon_t = \upsilon^i), \tag{24}$$

where

$$\pi_t(\upsilon^i) = \Pr(\upsilon_t = \upsilon^i | \mathcal{I}_t) \tag{25}$$

represents the subjective probability that the state is υ^i at time t and $u(C_t, t | \upsilon_t)$ an intertemporal utility. $\mathcal{I}_t$ represents the investor information set. The utility function $U(C_t, \upsilon_t)$ is called a belief-dependent utility function over a prize C_t.

Veronesi (2004) further proposes a parsimonious parametrization that is consistent with the external habit model of Campbell and Cochrane (1999). The utility function is given by

$$u(C_t, t | \upsilon_t) = e^{-\phi t} \frac{(C_t \upsilon_t)^{1-\alpha}}{1 - \alpha}, \tag{26}$$

where the surplus consumption ratio $s_t = \upsilon_t = \frac{C_t - X_t}{C_t}$, X_t is a slow-moving external habit, C_t represents consumption at date t, and ϕ is the subjective discount rate. Following Veronesi (2004), we assume that the surplus consumption ratio is perfectly correlated with the state υ_t. In this economy, the stochastic discount factor is

$$m_t^T = \prod_{\tau=t}^{T-1} m_{\tau+1}, \tag{27}$$

with

$$m_{\tau+1} = \frac{u_c(C_{t+1}, t+1 | \upsilon_{t+1})}{u_c(C_t, t | \upsilon_t)}, \tag{28}$$

where $u_c (C_t, t|v_t)$ is the marginal utility of consumption at time t. The marginal utility of consumption can be expressed as

$$u_c (C_t, t|v_t) = e^{-\phi t} k (v_t) C_t^{-\alpha},$$

with $k (v_t) = v_t^{-\alpha}$. Substituting this last expression in Equation (28) yields

$$m_{\tau+1} = e^{-\phi} \left(\frac{k (v_{\tau+1})}{k (v_\tau)} \right) \left(\frac{C_{\tau+1}}{C_\tau} \right)^{-\alpha}.$$

We will now assume that

$$\frac{k (v_{\tau+1})}{k (v_\tau)} = e^{-\rho(U_\tau)(\log(\frac{C_{\tau+1}}{C_\tau})-\log(\frac{C_\tau}{C_{\tau-1}}))}, \tag{29}$$

where U_t represents the state variable defined in Section 3. The state variable U_τ is different from v_τ. The parameter $\rho (U_\tau)$ can be interpreted as the state dependent measure of aversion to belief uncertainty. Note that in Veronesi (2004), the parameter $\rho (U_\tau)$ is not state-dependent. Given the specification in Equation (29), the stochastic discount factor $m_{\tau+1}$ can be written as

$$m_{\tau+1} = e^{-\phi} e^{-\rho(U_\tau)[\log(\frac{C_{\tau+1}}{C_\tau})-\log(\frac{C_\tau}{C_{\tau-1}})]} \left(\frac{C_{\tau+1}}{C_\tau} \right)^{-\alpha}. \tag{30}$$

Given Assumptions 1, 2, and 3′, we derive the price of a European call option in this model.

Proposition 4.2. *Under assumptions 1, 2, and 3′, the price of a European call option on a dividend-paying stock with strike K is given by the same formula as in Proposition 4.1, with*

$$\sum_{\tau=t}^{T-1} \mu_{m\tau+1} = -\phi(T - t) - (\rho(U_{T-1}) + \alpha)\mu_{X_T}$$

$$+ \sum_{\tau=t}^{T-2} [\rho (U_{\tau+1}) - (\rho (U_\tau) + \alpha)] \mu_{X_{\tau+1}} + \rho (U_t) \log \left(\frac{C_t}{C_{t-1}} \right),$$

$$\sum_{\tau=t}^{T-1} \sigma_{m\tau}^2 = (\rho (U_{T-1}) + \alpha)^2 \sigma_{X_T}^2 + \sum_{\tau=t}^{T-2} [\rho (U_{\tau+1}) - (\rho (U_\tau) + \alpha)]^2 \sigma_{X_{\tau+1}}^2,$$

$$\psi = \sum_{\tau=t}^{T-2} (\rho (U_{\tau+1}) - (\rho (U_\tau) + \alpha)) \sigma_{XY,\tau+1} - (\rho (U_{T-1}) + \alpha) \sigma_{XY,T}.$$

Proof. See Appendix. ∎

5. Empirical Illustrations

In this section, we pursue two main objectives. First, we want to illustrate that the models proposed in the previous section do indeed reproduce the puzzles for reasonable values of the model parameters. This will be the purpose of the numerical exercise conducted in Section 5.1. Second, we carry out a calibration exercise to attempt to discriminate between the recursive utility model with state-dependent preferences and the habit model with state-dependent beliefs. This is done in Section 5.2.

5.1 Numerical exercise

To illustrate the state-dependent economic models described in Section 4 and their effects on the puzzles put forward by Aït-Sahalia and Lo (2000); and Jackwerth (2000), we specify a two-state, first-order Markov process for the state variable U_t. We consider in turn a recursive utility model with state dependence in fundamentals and preferences and an external habit model with state dependence in beliefs. We produce graphs of the pricing kernel and the absolute-risk aversion as functions of discrete wealth states. These two functions are obtained using Equations (4) and (5). The objective and risk-neutral probabilities are computed according to the formulas developed in Section 3.2. For simplicity, we consider one-period options.

5.1.1 Recursive utility with state dependence in fundamentals and preferences.
We assume that both fundamentals and preferences can be affected by the latent state variables. The means and variances of the consumption and dividend growth rates take two values, one in each state. For the three preference parameters β, α, and ρ, we first assume that they all remain constant (Figure 1) and then that the risk aversion parameter α changes with the state while keeping the two other preference parameters fixed (Figure 2). We choose values of the parameters that are close to those estimated in Garcia, Luger, and Renault (2003), where preference parameters are not state-dependent.[12]

In Figure 1, we present the graphs for the risk-neutral and objective probabilities across wealth states and the resulting pricing kernel and absolute-risk aversion functions. Values for the parameters are specified in the legend of the figure. The left panel contains the marginal risk-neutral and objective distributions. It appears that the risk-neutral distribution has less mass in the center and the right-hand tail, making it more skewed to the left. The center panel in Figure 1 reveals that the unconditional pricing kernel increases in the center return states (over the range of 0.99 to 1.03). We use the term unconditional to emphasize that the pricing kernel function across wealth states is computed

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[12] Garcia, Luger, and Renault (2003) estimate a utility-based option pricing model in the recursive utility framework of Epstein and Zin (1989). They use daily price data for S&P 500 index European call options obtained from the Chicago Board Options Exchange for the period January 1991 to December 1995. They also use daily return data for the S&P 500 index and estimate preferences parameters in S&P 500 option prices. They find quite reasonable values for the coefficient of the risk aversion and the intertemporal elasticity of substitution.
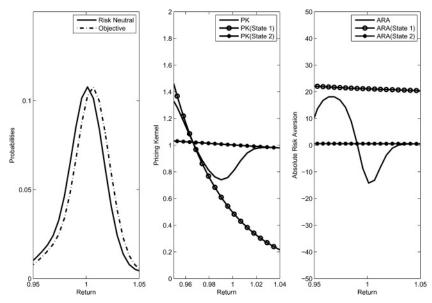
Figure 1
Risk-neutral and objective distributions, pricing kernel, and absolute-risk aversion functions with state dependence in fundamentals
The preference parameters are: $\beta = 0.98$, $\alpha = -5$, $\rho = -10$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0005, -0.0002)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0220, 0.0355)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0.001, 0.001)$, $\sigma_{Y_{t+1}} = (0.015, 0.12)$. The correlation coefficient between consumption and dividends is 0.15. The left graph contains the marginal objective and risk-neutral distributions based on Equations (12) and (19). The graph middle graph contains the corresponding pricing kernel (PK) functions. The last graph contains the absolute-risk aversion (ARA) functions across wealth states. The unconditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in Equations (12) and (19).

using marginal probabilities given by Equations (12) and (19). However, the pricing kernel in each regime (computed from the objective probabilities in Equation (9) and risk-neutral probabilities in Equation (16) declines monotonically across wealth states. On the right panel, for the center wealth states, we find as in Jackwerth (2000) that the risk aversion function becomes negative. Within each regime, the absolute-risk aversion functions across wealth states are perfectly decreasing functions of the aggregate wealth: the puzzle disappears.[13]

In Figure 2, we consider state dependence in both the fundamentals and the risk aversion parameter.[14] We maintain the same values for the model

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[13] Brown and Jackwerth (2000) propose as an explanation of the puzzles a consumption-based utility model where they introduce as an additional state variable, a momentum variable that follows a mean-reverting process. To produce the graphs of the pricing kernels, they simulate 10,000 runs of the processes based on a daily discretization over 31 days and smooth the graph by kernel regression. This corresponds in our setting to the marginal probabilities given by Equations (10) and (17), that is, the unconditional average of the probabilities given the trajectory of the state variables. For space considerations, we do not reproduce the corresponding graphs with our recursive utility model, since we obtain very similar patterns as in Figure 1.

[14] For space considerations, we do not report the case where the fundamentals remain constant and the preference parameters change with the state. The results obtained were very similar to the other reported cases. We also
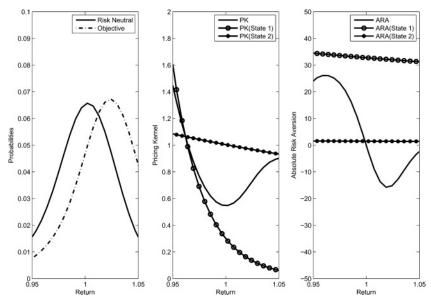
Figure 2
Pricing kernel and absolute-risk aversion functions with state dependence in fundamentals and preferences
The preference parameters are: $\beta = 0.99$, $\alpha = (-5, -2.5)$, $\rho = -10$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0005, -0.0002)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0220, 0.0355)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0.001, 0.001)$, $\sigma_{Y_{t+1}} = (0.025, 0.12)$. The correlation coefficient between consumption and dividends is 0.15. The left graph contains the marginal objective and risk-neutral distributions based on Equations (12) and (19). The middle graph contains the corresponding pricing kernel (PK) functions. The last graph contains the absolute-risk aversion (ARA) functions across wealth states. The unconditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in Equations (12) and (19).

parameters except for $\beta = 0.995$ and α, which now takes two values, -5 and -2.5. In the three panels, we observe patterns that are very similar to the respective ones drawn in Figure 1 where only the fundamentals change with the state. State-dependent risk aversion accentuates the difference between the objective and risk-neutral distributions with more weight being shifted to the left tail in the risk-neutral distribution. The regions over which the pricing kernel increases and the risk aversion becomes negative are displaced to the right and are larger in magnitude compared to Figure 1.

5.1.2 External habit with state dependence in beliefs. Campbell and Cochrane (1999) show that an external habit model is able to explain asset pricing puzzles, namely the equity premium puzzle, the risk-free rate puzzle, and the volatility puzzle. We show here that an extension of this model proposed

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investigated several state-dependent preference cases. First, we assumed a constant relative risk aversion (CRRA) and a state-dependent elasticity of intertemporal substitution (EIS). Second, we assumed a state-dependent risk aversion and a constant EIS. Third, we assumed cyclical CRRA and EIS, and finally we assumed state-dependent time preferences. For all combinations of state-dependent preference parameters, we get very similar results.
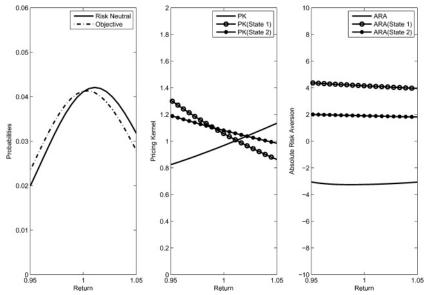
Figure 3
Pricing kernel and absolute-risk aversion functions with state dependence in fundamentals and beliefs
The preference parameters are: $\phi = 0.03$, $\alpha = 10$, $\rho = (10, 0)$. The regime probabilities are: $p_{11} = 0.9$, $p_{00} = 0.6$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.0015, -0.0019)$ and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.0159, 0.0390)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0)$, $\sigma_{Y_{t+1}} = (0.05, 0.12)$. The correlation coefficient between consumption and dividends is 0.45. The first graph contains the marginal objective and risk-neutral distributions based on equations (12) and (19). The middle graph contains the corresponding pricing kernel (PK) functions. The last graph contains the absolute-risk aversion (ARA) functions across wealth states. The unconditional ARA (PK) function is the ARA (PK) function computed with the objective and risk-neutral probabilities in Equations (12) and (19).

by Veronesi (2004) and set in a state-dependent framework can explain the risk aversion puzzle demonstraded by Aït-Shalia and Lo (2000); and Jackwerth (2000) based on prices on index options and returns on the index. This extension has the virtue of capturing state-dependent beliefs, since Ziegler (2007) has advocated heterogeneity of beliefs as an explanation for the puzzles.

In our calibration, we keep the parameters of the fundamentals close to the ones we have used in the examples in the previous section on recursive utility. We have increased slightly the volatility parameters as well as the correlation coefficient between dividends and consumption. For the preference parameters, we start from the values used in Veronesi (2004). Since the fundamentals have different dynamics, we adjust upward the risk aversion parameter (10 instead of 1.5 to 3). However, we reduce the magnitude of the ρ parameter. The parameter ϕ is set at 0.03, close to the value chosen in Veronesi (2004). The three panels in Figure 3 display, from left to right, the objective and risk-neutral probability distributions, the unconditional and conditional pricing kernels, and the unconditional and conditional absolute-risk aversion functions, respectively. Although we also reproduce the puzzles with state-dependent beliefs, the graphs

are strikingly different from the ones obtained with recursive utility. The wealth regions over which the pricing kernel function remains increasing and the risk aversion function positive are much wider than in the recursive utility case. An interesting feature is that aversion to state uncertainty inverts the position of the risk-neutral and objective distributions with respect to the recursive utility case. The risk-neutral distribution is shifted to the right as in the post-1987 period (Figure 2 in Jackwerth, 2000). Also, both distributions have much less mass in the center compared with the recursive utility model. With such a difference between the two models, we should be able to discriminate between the two models in the data. This is the goal of the calibration exercise in the next section.

5.2 Application to S&P 500 data

Evidence about investors' preferences inferred from asset prices is aggregate in nature. As emphasized in the introduction, there is considerable variation over time in the estimates of the risk aversion function. The equilibrium models with state dependence that we proposed in the previous section can accommodate changing behavioral characteristics of the representative investor. Their main merit is to make explicit which preference or belief parameter changes with the state. Short of estimating and formally testing these models,[15] we will calibrate the various models to the S&P 500 index and option data to assess whether changing beliefs or preferences are more likely to explain the risk aversion and pricing kernel puzzles.

We will base our calibration on the data used in Jackwerth (2000). Starting with a database that contains minute-by-minute trades and quotes covering S&P 500 European index options and S&P 500 index futures, Jackwerth (2000) gathers 114 daily sets of 31-day index return and 1-month maturity options over the 1987 to 1995 period. The resulting nonoverlapping 31-day risk aversion functions are further averaged for several subperiods to illustrate the risk aversion puzzle. Whether one looks at these averages over several years, or at averages over each year or at the daily risk aversion functions, it is clear that the shapes vary considerably over time. We illustrate this in Section 5.2.1. In two specific days, illustrative of specific economic circumstances, we find support for each model in turn. To arrive at an overall evaluation of the models with changing preferences or changing beliefs, we proceed with a matching of the yearly averages of the pricing kernel and risk aversion functions. The results are presented in Section 5.2.2.

5.2.1 The distribution of the absolute-risk aversion and pricing kernel functions. In Figure 4, we plotted the average pricing kernels and risk aversion functions over the years 1988, 1991, 1993, and 1995. The puzzles are

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[15] In Garcia, Luger, and Renault (2003), we formally estimate a restricted version of the recursive utility model presented in Section 4.1 where only fundamentals change with the state.
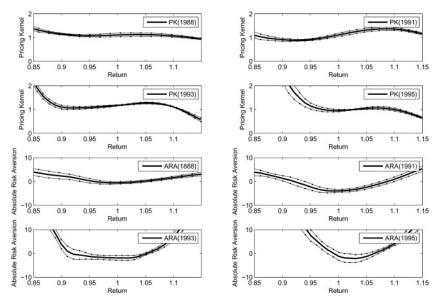
Figure 4
Observed pricing kernels and absolute-risk aversion functions for the S&P 500 index
Every 31 days from April 1986 to December 1995, the objective and risk-neutral distributions for the 31-day index return are calculated with the methodologies used in Jackwerth (2000). A return (wealth) level of one corresponds to the forward price of the index, where the forward contract has the same maturity as the options used for obtaining the risk-neutral probability distribution. For each wealth level, the mean pricing kernel and absolute-risk aversion functions are calculated across the years 1988, 1991, 1993, and 1995. Based on the empirical standard deviations of the two measures, we also draw 5% confidence intervals.

present in all graphs, but the shapes appear very different from one year to the other. In the wealth ranges where marginal utility is increasing, the steepness of the slopes of the pricing kernels varies markedly. Similarly, the wealth range over which we observe a negative risk aversion changes from one year to the other depending on the curvature of the U-shaped function. We will calibrate the models over these changing shapes in the next section, but first we want to show that the two models appear as good candidates, since each can reproduce the shapes observed in one of the 114 daily functions. One is chosen in December 1987, two months after the great October 1987 stock market crash, where heterogeneity in risk aversion is more likely to prevail. We contrast this day with a day selected in February 1990, just before the recession that started in July 1990. It was certainly a period with dispersed beliefs about the future course of the economy and financial markets. Indeed, even *ex-post*, many causes have been cited for the 1990–1991 recession, such as consumer pessimism, debt accumulation of the 1980s, high oil prices, credit crunch, and monetary policy actions to curb inflation.[16]

The two left-hand panels of Figure 5 display the results of matching, on 15 December 1987, the observed pricing kernel to the recursive utility model

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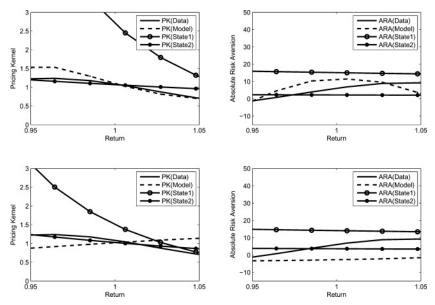[16] See for example Walsh (1993) for an investigation of the causes of the 1990–1991 recession.

Figure 5
Observed and calibrated pricing kernel and absolute-risk aversion functions on 15 December 1987
This graph displays the pricing kernel and absolute-risk aversion functions implied by the data on 15 December 1987 as well as the corresponding calibrated functions inferred from two equilibrium option pricing models. The two top panels correspond to the recursive utility model with state dependence in fundamentals and preferences, while the two bottom ones are based on the external habit model with state dependence in fundamentals and beliefs. The objective and risk-neutral distributions for the 31-day index return are calculated with the methodologies used in Jackwerth (2000). The pricing kernel function across wealth is calculated as the ratio of the risk-neutral probability to the objective probability. The absolute-risk aversion function across wealth is calculated as the negative of the first derivative of the pricing kernel function.

with state-dependent fundamentals and risk aversion (top), and an external habit model with state-dependent beliefs (bottom). In each panel, we plot the pricing kernel function observed in the data, the unconditional function, and the pricing kernels in the two states. The calibration proceeded by creating grids around starting values for each parameter. The starting values for the fundamentals are based on values close to the ones estimated in Garcia, Luger, and Renault (2003). The values that produced the closest pricing kernels to the empirical ones over the 0.95–1.05 wealth range are reported in Table 1. It appears that the state-dependent recursive utility model reproduces better the slightly downward and curved pattern of the pricing kernel on that particular day. The belief model produces a flatter line with a positive slope. For the risk aversion functions on the two right-hand panels, it is also the case that the recursive utility model is closer to the data-produced function than the belief model, but it seems to produce too much curvature compared to the empirical function.

This is what makes this model perform less satisfactorily when another date is chosen, 13 February 1990. The corresponding functions are displayed in

Table 1
Calibration parameters

Panel A

State dependence in fundamentals and preferences

β	ρ	p_{11}	p_{22}	ρ_{XY}	α		$\mu_{X_{t+1}}$	$\sigma_{X_{t+1}}$	$\mu_{Y_{t+1}}$	$\sigma_{Y_{t+1}}$
0.999	−8.5	0.92	0.895	0.3	$U_t = 1$	−1.5	0.00035	0.0159	0.000	0.03
					$U_t = 2$	−8	−0.0025	0.0250	0.025	0.10

State dependence in beliefs

ϕ	α	p_{11}	p_{22}	ρ_{XY}	ρ		$\mu_{X_{t+1}}$	$\sigma_{X_{t+1}}$	$\mu_{Y_{t+1}}$	$\sigma_{Y_{t+1}}$
0.020	10	0.950	0.500	0.45	$U_t = 1$	1.50	0.0025	0.0159	0.000	0.050
					$U_t = 2$	−1.50	−0.0019	0.0341	0.000	0.120

Panel B

State dependence in fundamentals and preferences

β	ρ	p_{11}	p_{22}	ρ_{XY}	α		$\mu_{X_{t+1}}$	$\sigma_{X_{t+1}}$	$\mu_{Y_{t+1}}$	$\sigma_{Y_{t+1}}$
0.999	{−9.5, −9, 8.5}	0.92	0.895	0.3	$U_t = 1$	{−1.5, −2}	0.00035	0.0159	0.000	0.03
					$U_t = 2$	{−9.5, −9, −8.5}	−0.0025	0.0250	0.025	0.10

State dependence in beliefs

ϕ	α	p_{11}	p_{22}	ρ_{XY}	ρ		$\mu_{X_{t+1}}$	$\sigma_{X_{t+1}}$	$\mu_{Y_{t+1}}$	$\sigma_{Y_{t+1}}$
0.020	{9, 9.5, 10}	0.950	0.500	0.45	$U_t = 1$	{1.50, 2}	0.0025	0.0159	0.000	0.050
					$U_t = 2$	{−1.50, −2}	−0.0019	0.0341	0.000	0.120

This table displays the values of preference, belief, and fundamentals parameters used to calibrate the various option pricing models. Figures 5 and 6 are obtained with preference parameters appearing in panel A. Figure 7 is obtained by disturbing the preference parameters in a reasonable range and holding fixed the fundamental parameters. In panel B, we report only the set of parameters that best describe the ARA and PK functions given the dataset.

Figure 6. On that particular date, the empirical pricing kernel and risk aversion functions are flatter and are much better matched by the state-dependent belief model. When we look at both functions in each of the two states, the shapes are similar, but the levels differ considerably between the recursive utility with state-dependent risk aversion and the habit model with state-dependent beliefs.

The fact that the recursive utility model is a better match than the belief model just coming out of the 1987 crisis may not be too surprising. A higher risk aversion with a sizable weight could better represent the average investor, since the beliefs may have been more or less uniform at that time. This was not the case in 1990. In the next section, we proceed to a more systematic comparison of the two models at an annual frequency over the eight years of our sample.

5.2.2 Calibration of the models over one-year periods. It is hard to decide at which frequency one would like to evaluate the equilibrium models. We have at our disposal 114 daily sets of futures returns and option prices at a maturity of one month. Our goal is to assess whether the evidence is tilted more toward one model than the other in some average sense. Given the evidence put forward in Section 5.2.1 about the diversity of shapes of the yearly empirical pricing kernel and risk aversion functions, we will match the models with respect to the yearly empirical functions.

We follow the same grid-search calibration procedure over the preference or belief parameters to match as closely as possible the empirical functions over the 0.95–1.05 range for wealth. We conduct this exercise for each of the eight
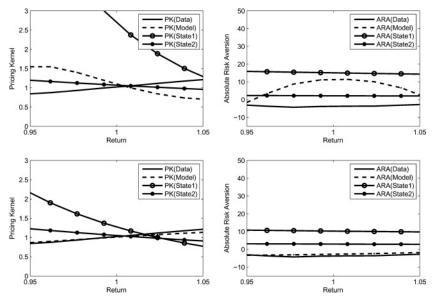
Figure 6
Observed and calibrated pricing kernel and absolute-risk aversion functions on 13 February 1990
This graph displays the pricing kernel and absolute-risk aversion functions implied by the data on 13 February 1990 as well as the corresponding calibrated functions inferred from two equilibrium option pricing models. The two top panels correspond to the recursive utility model with state dependence in fundamentals and preferences, while the two bottom ones are based on the external habit model with state dependence in fundamentals and beliefs. The objective and risk-neutral distributions for the 31-day index return are calculated with the methodologies used in Jackwerth (2000). The pricing kernel function across wealth is calculated as the ratio of the risk-neutral probability to the objective probability. The absolute-risk aversion function across wealth is calculated as the negative of the first derivative of the pricing kernel function.

years between 1988 and 1995. In Figure 7, we draw the same four panels as in Figures 5 and 6 but this time for the year 1994, representative of all years in the sample.

The state-dependent belief habit model fits closer than the recursive utility model for both the pricing kernel and the risk aversion function. With state-dependent risk aversion in the recursive utility model, the slope of the pricing kernel is negative while it is positive in the data. The risk aversion function produced by this model is also counterfactual. The negative values occur in the wealth range before 0.96, whereby in the data, the risk aversion is positive. The reverse is true above the 0.96 value: the empirical risk aversion is negative, while the recursive utility model function is positive.

This is certainly not the case for the belief model, as evidenced in the bottom right panel. The risk aversion functions of the data and of the model are very close to each other and have similar slopes. For space considerations, we do not report the same figures for all years since they look pretty similar to Figure 7. While the slopes differ from one year to the next, the belief model captures well the general pattern. It should be emphasized that the habit model fits as well in the recession years of 1990 and 1991.
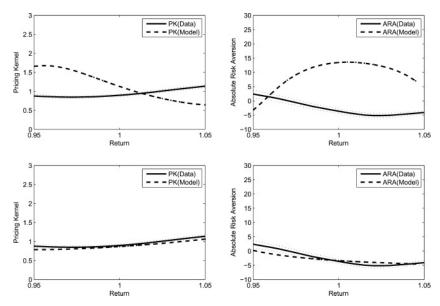
Figure 7
Observed and calibrated pricing kernel and absolute-risk aversion functions in 1994
This graph displays the average pricing kernel and absolute-risk aversion functions implied by the twelve 31-day return and option datasets in the year 1994, with their empirical 5% confidence intervals as well as the corresponding calibrated functions inferred from two equilibrium option pricing models. The two top panels correspond to the recursive utility model with state dependence in fundamentals and preferences, while the two bottom ones are based on the external habit model with state dependence in fundamentals and beliefs. The objective and risk-neutral distributions for the 31-day index return are calculated with the methodologies used in Jackwerth (2000). The pricing kernel function across wealth is calculated as the ratio of the risk-neutral probability to the objective probability. The absolute-risk aversion function across wealth is calculated as the negative of the first derivative of the pricing kernel function.

Hence, based on this yearly calibration, we tend to favor the habit model with state-dependent beliefs over the recursive utility model with state-dependent risk aversion. Of course, the fact that we averaged over a year the twelve 30-day interval functions may be the reason behind this result.[17] This suggests that a more thorough estimation exercise should be conducted to test these models in order to better locate the source of heterogeneity among investors over time.

6. Discussion

As risk-neutral expectations of discounted terminal payoffs, option prices are only informative about the risk-neutral distribution. For example, in Renault (1997), it is shown that, in the presence of a leverage effect, conditioning by the future volatility path has an effect on returns forecasts and implies a very steep and asymmetric volatility smile. However, the volatility smile

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[17] In Garcia, Luger, and Renault (2003), where we estimated the model every week in a rolling window fashion, we were able to find evidence in support of the recursive utility model over the simpler expected utility model, but with constant preference parameters.

does not provide information on the potential difference between p and p^*. As we have emphasized in this paper, any difference is well captured by odd shapes of the pricing kernel or the risk aversion function. Therefore, any model that aims at explaining these puzzles must make clear why it captures the relevant differences between p and p^*. Three main approaches can be found in the literature. The difference between the risk-neutral and objective skewness after 1987 has been invoked by Jackwerth (2000) to justify the pricing kernel puzzle. However, one needs to relate the risk-neutral skewness to the objective skewness and explain why they differ. A second line of explanation is based on the fact that p^* is forward looking, while p is backward looking. But one must explain why this implies a difference between the risk-neutral and the objective probabilities in a stationary world. A final potential source of the puzzles has been associated with estimation error, but one may ask why we should observe a persistent discrepancy between p and p^* even when the number of observations is large.

In this paper, we provide a unifying framework that gives content to all these explanations. To link the risk-neutral skewness and the objective skewness, we refer to Bakshi, Kapadia, and Madan (2003). A way to capture this divergence between the risk-neutral and the physical distributions is through their higher-order moments. Bakshi, Kapadia, and Madan show that, in power utility economies (with a pricing kernel exponential in the index returns, $e^{-\gamma R_m}$), the risk-neutral skewness of the index returns is linked approximately to the higher moments of the physical distribution, denoted by an overline, by the following relation:

$$Skew(t, t + \tau) \approx \overline{Skew}(t, t + \tau) - \gamma(\overline{Kurt}(t, t + \tau) - 3)\overline{Std}(t, t + \tau), \quad (31)$$

where γ denotes the coefficient of relative risk aversion.[18] Therefore, fat tails in the physical distribution will produce a more negatively skewed risk-neutral distribution. Clark (1973) explains that the lack of conditioning implies excess kurtosis. Therefore, the unconditional distribution, which is a mixture of the conditionals upon the state variables distributions, will exhibit fat tails.

According to Ziegler (2007), nonstationarity of the return process may explain that estimates of beliefs obtained from historical frequency return distributions will differ from agents' actual assessments. The presence of the state variables in our setting introduces what Ziegler calls nonstationarity. The observation of the past values of the state variables modifies the expectation about the future values of the state variables and in turn about future returns. Ziegler (2007) concludes that the beliefs are heterogeneous and that the researcher ignores belief heterogeneity. This is observationally equivalent to our state

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[18] Bakshi, Kapadia, and Madan (2003) show that this relation can be generalized to a broader family of utility functions. In particular, in the case of state-dependent utilities, the skew dynamics can depend on conditional moves in risk aversion. Bates (2000) presents evidence that post-1987 distributions implicit in S&P 500 futures options are strongly negatively skewed.

variable framework. Moreover, we provide a structural model where the beliefs depend on the state variables and which can be estimated once a stochastic process is assumed for the state variables.

Ziegler (2007) also argues that it is hard to rationalize the implied risk aversion smile by belief misestimation if agents have homogeneous beliefs. We have seen that it can easily be rationalized by the difference between estimating the unconditional distribution of returns and the conditional one, given state variables. In particular, one cannot exclude the possibility of state dependence in the fundamentals, which will produce similar effects on the estimate of the risk aversion function.

We invoke causality arguments to rationalize the observed puzzles in terms of conditioning. Let $m(U_t^T, \frac{C_T}{C_t})$ be the SDF that prices payoffs between t and T. Even though the function $c \to m(u, c)$ is a well-decreasing function in c, one may find increasing patterns when simulating paths of (u, c) to obtain a kernel smoother of $E_t[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T]$. This is what Brown and Jackwerth (2000); and Ziegler (2007) do. The former simulates wealth together with a momentum process, while the latter simulates Pan's (2002) model. It should be emphasized that the puzzle can only occur because C_T is not independent from U_t^T, which implies that

$$
E_t\left[m\left(U_t^T, \frac{C_T}{C_t}\right)\middle| C_T = c_T\right] \neq E_t\left[m\left(U_t^T, \frac{c_T}{C_t}\right)\right]. \tag{32}
$$

Otherwise, a decreasing $c \to m(u, c)$ will imply that $c \to E[m(u, c)]$ will also be decreasing. In our general setting of pricing with state variables, $E_t[m(U_t^T, \frac{C_T}{C_t})|C_T = c_T]$ is not always decreasing, while $c \to m(u, c)$ is always decreasing.

Even though our framework is set with exogenous state variables, the odd patterns in the pricing kernel and the risk aversion functions can be obtained in the model of Heston and Nandi (2000) for more than one-period options.[19] The intuition is that a GARCH option pricing SDF aggregated over several periods has similarities with a state-dependent SDF.

To rationalize the fact that the pricing kernel puzzle is not maintained when using a proper conditioning, observe that

$$
E_t\left[m\left(U_t^T, \frac{C_T}{C_t}\right)\middle| C_T = c_T, U_1^t\right] = E_t\left[m\left(U_t^T, \frac{c_T}{C_t}\right)\middle| U_1^t\right], \tag{33}
$$

since C does not cause the state variables U by Assumption 2 in a consumption model. Therefore, the decreasing feature of $c \to m(u, c)$ is not modified by the expectation operator.

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[19] We derive the expressions for the risk-neutral and objective distributions and produce graphs similar to the ones in Section 5. They are available upon request from the authors.

7. Conclusion

The main goal of this paper was to reconcile with economic theory the puzzling facts about the pricing kernel and the risk aversion functions extracted from option and equity prices. Our central contribution has been to provide a unifying explanation in terms of state dependence when the states may be observed by the investors but not by the researcher. Since the goal of this line of research is to better identify investors' preferences or beliefs, we provide option pricing formulas for several economic models with state dependence. These formulas allow us to recover analytically the risk-neutral and objective probabilities across wealth states, and thus, the risk aversion and pricing kernel functions. We show that models with regime shifts in fundamentals, preferences, or beliefs can rationalize the puzzles put forward in Aït-Sahalia and Lo (2000) and Jackwerth (2000). The absolute-risk aversion and pricing kernel functions extracted from calibrated prices in these economies exhibit the same puzzling features as in the original papers and are inconsistent with the usual assumptions of decreasing marginal utility and positive risk aversion. However, the investor utility is well-behaved and her risk aversion remains positive. In other words, investors' behavior is not at odds with economic theory, but depends on some factors that the statistician does not observe.

Appendix

Proof of Proposition 3.1. For space considerations, since the proofs are very similar, we refer to the main proof below of Propositions 4.1 and 4.2 for the various steps to be followed to prove the proposition. A fully developed proof is available from the authors upon request. ∎

To give a formal proof of Propositions 4.1 and 4.2, we first consider the following two lemmas:

Lemma 1. *Under Assumptions 1, 2, and 3′, the conditional probability distribution of* $(\log m_{t+1}, \log \frac{S_{t+1}}{S_t})$ *given* U_1^{t+1} *is jointly normal.*

Proof of Lemma 1.

Case 1. *Recursive Utility with State Dependence in Fundamentals and Preferences*
    We have

$$\begin{bmatrix} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{bmatrix} = A + B \begin{bmatrix} \log \frac{C_{t+1}}{C_t} \\ \log \frac{D_{t+1}}{D_t} \end{bmatrix}, \tag{A1}$$

where $A = (a_1, a_2)'$ with

$$a_1 = \gamma(U_t) \log \beta(U_t) + \left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})} \right) \log \left(\frac{\lambda\left(U_1^{t+1}\right) + 1}{\lambda(U_t)} \right)$$

$$+ \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - \frac{\alpha(U_t)}{\rho(U_t)} \right) \log(\lambda(U_t) C_t),$$

$$a_2 = \log \frac{\varphi\left(U_1^{t+1}\right)}{\varphi\left(U_t\right)},$$

and B is a diagonal matrix with diagonal coefficients

$$b_{11} = \left[\gamma(U_t)\left(\rho(U_t) - \frac{\rho(U_t)}{\rho(U_{t+1})}\right) + \left(\frac{\alpha(U_t)}{\rho(U_{t+1})} - 1\right)\right],$$

$$b_{22} = 1.$$

Using Assumption 3′, it is straightforward to show

$$\begin{bmatrix} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{bmatrix} \bigg/ U_1^{t+1} \rightsquigarrow N[\mu, \Sigma_{ms}],$$

with

$$\mu = A + B \begin{pmatrix} \mu_{X_{t+1}} \\ \mu_{Y_{t+1}} \end{pmatrix}, \qquad \Sigma_{ms} = B \begin{pmatrix} \sigma^2_{X_{t+1}} & \sigma_{X_{t+1}Y_{t+1}} \\ \sigma_{X_{t+1}Y_{t+1}} & \sigma^2_{Y_{t+1}} \end{pmatrix} B'.$$

Case 2. *External Habit Model with State Dependence in Beliefs*

We have

$$\begin{bmatrix} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{bmatrix} = A + B \begin{bmatrix} \log \frac{C_{t+1}}{C_t} \\ \log \frac{D_{t+1}}{D_t} \end{bmatrix},$$

where

$$A = \begin{bmatrix} -\phi + \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right) \\ \log \frac{\varphi\left(U_1^{t+1}\right)}{\varphi(U_t)} \end{bmatrix} \text{ and } B = \begin{bmatrix} -(\alpha + \rho(U_t)) & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Assumption 3′, it is straightforward to show that the joint process $(\log m_{t+1}, \log \frac{S_{t+1}}{S_t})$ is normally distributed. ∎

Lemma 2. *Assume that the random variable (Z_1, Z_2) is normally distributed:*

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \rightsquigarrow N\left(\begin{pmatrix} \mu_{Z_1} \\ \mu_{Z_2} \end{pmatrix}, \begin{bmatrix} \sigma^2_{Z_1} & \rho_{Z_1,Z_2}\sigma_{Z_1}\sigma_{Z_2} \\ \rho_{Z_1,Z_2}\sigma_{Z_1}\sigma_{Z_2} & \sigma^2_2 \end{bmatrix}\right).$$

Let denote Q, the probability measure corresponding to the process (Z_1, Z_2) and Q^*, the probability measure defined by

$$\frac{dQ^*}{dQ}(Z) = \exp\left[(Z - \mu_{Z_1}) - \frac{\sigma^2_{Z_1}}{2}\right].$$

Then, by the Girsanov theorem,

$$Q^*(Z_2 \geq 0) = 1 - \Phi\left[\frac{-\mu_{Z_2} - \rho_{Z_1,Z_2}\sigma_{Z_1}\sigma_{Z_2}}{\sigma_{Z_2}}\right]$$

$$= \Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_1,Z_2}\sigma_{Z_1}\right] \qquad (A2)$$

and

$$E\left[\exp\left(Z_1\right)1_{Z_2\geq0}\right] = \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right]\Phi\left[\frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_1,Z_2}\sigma_{Z_1}\right]$$

$$= \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right]Q^*\left(Z_2 \geq 0\right),$$

where Φ is the cumulative normal distribution function.

Proof of Propositions 4.1 and 4.2. At time t, a European call option on the dividend-paying stock with strike K that expires at T is worth

$$\pi_t = E_t\left[m_t^T \max\left(S_T + D_T - K, 0\right)\right],$$

with

$$m_t^T = \prod_{\tau=t}^{T-1} m_{\tau+1}.$$

First, we show that $\pi_t = E_t[S_t Q_{XY}(t,T)\Phi(d_1(x_t)) - K\widetilde{B}(t,T)\Phi(d_2(x_t))]$.

Let denote U_1^T the trajectory of state variables from t up to T, so that

$$\frac{\pi_t}{S_t} = E_t\left[m_t^T \max\left(S_T + D_T - K, 0\right)\right]$$

$$= E_t\left[E_t\left[m_t^T \max\left(S_T + D_T - K, 0\right)|U_1^T\right]\right]$$

$$= E_t\left[H\left(U_t^T\right) - \frac{K}{S_t}G\left(U_t^T\right)\right],$$

with

$$H\left(U_t^T\right) = E_t\left[m_t^T\left(\frac{S_T + D_T}{S_t}\right)\mathbf{1}_{\frac{S_T+D_T}{S_t} > \frac{K}{S_t}}\Big|U_1^T\right],$$

$$G\left(U_t^T\right) = E_t\left[m_t^T\mathbf{1}_{\frac{S_T+D_T}{S_t} > \frac{K}{S_t}}|U_1^T\right].$$

We denote

$$Z_1 = \log m_t^T = \sum_{\tau=t}^{T-1} \log m_{\tau+1} \text{ and } Z_2 = \log\frac{S_T + D_T}{S_t} - \log\frac{K}{S_t}. \tag{A3}$$

Under Assumptions 1, 2, and 3′, Lemma 1 can be used to show that the joint process (Z_1, Z_2) is conditionally normally distributed. We then use Lemma 2 to deduce

$$G\left(U_t^T\right) = E_t\left[m_t^T\mathbf{1}_{\frac{S_T+D_T}{S_t} > \frac{K}{S_t}}|U_1^T\right] = \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right]Q^*\left(Z_2 \geq 0\right)$$

$$= \widetilde{B}\left(t,T\right)\Phi\left(d_2\left(x_t\right)\right),$$

where

$$\widetilde{B}\left(t,T\right) = \exp\left[\mu_{Z_1} + \frac{\sigma_{Z_1}^2}{2}\right] \text{ and } d_2\left(x_t\right) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_1,Z_2}\sigma_{Z_1},$$

with

$$\mu_{Z_1} = E_t\left[Z_1|U_1^T\right] \text{ and } \sigma_{Z_1}^2 = Var_t\left[Z_1|U_1^T\right],$$
$$\mu_{Z_2} = E_t\left[Z_2|U_1^T\right] \text{ and } \sigma_{Z_2}^2 = Var_t\left[Z_2|U_1^T\right].$$

It is worth noticing that

$$B(t,T) = E_t\widetilde{B}(t,T).$$

Under Assumptions 1, 2, and 3′, Lemma 1 can be used to show that the joint-process (Z_2,Z_3) is conditionally normally distributed. We use Lemma 2 to deduce

$$H\left(U_t^T\right) = E_t\left[m_t^T\left(\frac{S_T+D_T}{S_t}\right)\mathbf{1}_{\frac{S_T+D_T}{S_t} > \frac{K}{S_t}}\bigg| U_1^T\right]$$
$$= \exp\left[\mu_{Z_3} + \frac{\sigma_{Z_3}^2}{2}\right]Q^*(Z_2 \geq 0),$$

where

$$\mu_{Z_3} = E_t\left[Z_3|U_1^T\right] \text{ and } \sigma_{Z_3}^2 = Var_t\left[Z_3|U_1^T\right],$$

with

$$Z_3 = \log\left(m_t^T\left(\frac{S_T+D_T}{S_t}\right)\right).$$

Since

$$d_1(x_t) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_2,Z_3}\sigma_{Z_3}, \text{ and } Q_{XY}(t,T)$$
$$= E_t\left[m_t^T\left(\frac{S_T+D_T}{S_t}\right)|U_1^T\right] = \exp\left[\mu_{Z_3} + \frac{\sigma_{Z_3}^2}{2}\right],$$

we deduce

$$H\left(U_t^T\right) = Q_{XY}(t,T)\Phi(d_1(x_t)).$$

Consequently, the price of a European call option is

$$\pi_t = E_t\left[S_t Q_{XY}(t,T)\Phi(d_1(x_t)) - K\widetilde{B}(t,T)\Phi(d_2(x_t))\right].$$

Second, we expand the quantities $Q_{XY}(t,T)$, $d_1(x_t)$, and $d_2(x_t)$.

Notice that under Assumptions 1, 2, and 3′, $Q_{XY}(t,T)$ can be rewritten as

$$Q_{XY}(t,T) = \exp\left(E_t\left(\log m_t^T|U_1^T\right) + \frac{1}{2}Var_t\left(\left(\log m_t^T|U_1^T\right)\right)\right)$$
$$\times \exp\left(E_t\log\left(\frac{S_T+D_T}{S_t}\right)\bigg|U_1^T + \frac{1}{2}Var_t\right.$$
$$\times\left.\left(\log\left(\frac{S_T+D_T}{S_t}\right)\bigg|U_1^T\right)\right)$$
$$\times \exp\left(Cov_t\left(\log m_t^T, \log\left(\frac{S_T+D_T}{S_t}\right)\bigg|U_1^T\right)\right).$$

Hence

$$Q_{XY}(t,T) = \left(E_t\left[m_t^T | U_1^T\right]\right)\left(E_t\left(\frac{S_T + D_T}{S_t}\bigg| U_1^T\right)\right)\exp(\psi)$$

$$= \widetilde{B}(t,T)E_t\left(\frac{S_T + D_T}{S_t}\bigg| U_1^T\right)\exp(\psi),$$

with

$$E_t\left[m_t^T | U_1^T\right] = \exp\left[\mu_{z_1} + \frac{\sigma_{Z_1}^2}{2}\right] = \widetilde{B}(t,T) \quad \text{and} \quad \psi$$

$$= Cov_t\left(\log m_t^T, \log\left(\frac{S_T + D_T}{S_t}\right)\bigg| U_1^T\right).$$

Given expressions appearing in (A3), we rewrite $d_2(x_t)$ as

$$d_2(x_t) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_1,Z_2}\sigma_{Z_1}$$

$$= \frac{E_t\left(\log\frac{S_T+D_T}{S_t}|U_1^T\right) - \log\frac{K}{S_t} + Cov\left(\sum_{\tau=t}^{T-1}\log m_{\tau+1}, \log\frac{S_T+D_T}{S_t}|U_1^T\right)}{\sqrt{Var_t\left(\log\frac{S_T+D_T}{S_t}|U_1^T\right)}}.$$

Since

$$\log Q_{XY}(t,T) = +\left(E_t\log\left(m_t^T\right)|U_1^T + \frac{1}{2}Var_t\left(\log\left(m_t^T\right)|U_1^T\right)\right)$$

$$+ \left(E_t\log\left(\frac{S_T + D_T}{S_t}\right)\bigg| U_1^T + \frac{1}{2}Var_t\right.$$

$$\times \left.\left(\log\left(\frac{S_T + D_T}{S_t}\right)\bigg| U_1^T\right)\right)$$

$$+ Cov_t\left(\log m_t^T, \log\left(\frac{S_T + D_T}{S_t}\right)\bigg| U_1^T\right)$$

and

$$Var_t\left(\log\left(\frac{S_T + D_T}{S_t}\right)\bigg| U_1^T\right) = \sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2},$$

$d_2(x_t)$ can be rewritten as

$$d_2(x_t) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_1,Z_2}\sigma_{Z_1}$$

$$= \frac{x_t + \log\left(Q_{XY}(t,T)\frac{B(t,T)}{\widetilde{B}(t,T)}\right) - \frac{1}{2}\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}{\sqrt{\sum_{\tau=t}^{T-1}\sigma_{Y\tau+1}^2}}.$$

In addition, notice that

$$Z_3 = Z_1 + Z_2 + \log\frac{K}{S_t},$$

and

$$\rho_{Z_2,Z_3}\sigma_{Z_3}\sigma_{Z_2} = Cov\,(Z_3,\,Z_2) = \rho_{Z_1,Z_2}\sigma_{Z_1}\sigma_{Z_2} + \sigma_{Z_2}^2.$$

Thus, $d_1\,(x_t)$ can be rewritten as

$$d_1\,(x_t) = \frac{\mu_{Z_2}}{\sigma_{Z_2}} + \rho_{Z_2,Z_3}\sigma_{Z_3} = d_2\,(x_t) + \sigma_{Z_2}.$$

Third, we show that (Z_1, Z_2), *and* (Z_2, Z_3) *are normally distributed, and give expressions for the coefficients* ψ, μ_{z_1}, *and* $\sigma_{Z_1}^2$ *when the pricing kernels are expressed as in Equation (21) or (30).*

Case 1. *When the pricing kernels are expressed as in Equation (21).*
   We first find the distribution of Z_1 and Z_2. Denote

$$
\begin{aligned}
\psi_1 = {}& \sum_{\tau=t}^{T-1} \gamma\,(U_\tau) \log\,(\beta\,(U_\tau)) + \sum_{\tau=t}^{T-1} \left(\frac{\alpha\,(U_\tau)}{\rho\,(U_{\tau+1})} - \gamma\,(U_\tau) \right) \log(C_t) \\
& + \sum_{\tau=t}^{T-1} \left(\frac{\alpha\,(U_\tau)}{\rho\,(U_{\tau+1})} - 1 \right) \log \left(\frac{\lambda\left(U_1^{\tau+1} \right) + 1}{\lambda\,(U_1^\tau)} \right) \\
& + \sum_{\tau=t}^{T-1} \left(\frac{\alpha\,(U_\tau)}{\rho\,(U_{\tau+1})} - \gamma\,(U_\tau) \right) \log\,(\lambda\,(U_1^\tau)) ,
\end{aligned}
\tag{A4}
$$

and

$$\psi_2 = \log \frac{\left(\varphi\,(U_1^T) + 1 \right)}{\varphi\left(U_1^{T-1} \right)} + \sum_{\tau=t}^{T-2} \log \frac{\varphi\left(U_1^{\tau+1} \right)}{\varphi\,(U_1^\tau)} - \log \frac{K}{S_t}. \tag{A5}$$

Under Assumption 3$'$, given the state variable trajectory, the random variable is

$$
\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \sum_{\tau=t}^{T-1} \begin{bmatrix} a_{1\tau+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \log \frac{C_{\tau+1}}{C_\tau} \\ \log \frac{D_{\tau+1}}{D_\tau} \end{bmatrix} + \sum_{\tau=t+1}^{T-1} \begin{bmatrix} a_{0\tau+1} \\ 0 \end{bmatrix} \sum_{\iota=t}^{\tau-1} \log \frac{C_{\iota+1}}{C_\iota},
$$

with

$$a_{0\tau+1} = \frac{\alpha\,(U_\tau)}{\rho\,(U_{\tau+1})} - \gamma\,(U_\tau),$$

$$a_{1\tau+1} = \gamma\,(U_\tau) \left(\rho\,(U_\tau) - \frac{\rho\,(U_\tau)}{\rho\,(U_{\tau+1})} \right) + \left(\frac{\alpha\,(U_\tau)}{\rho\,(U_{\tau+1})} - 1 \right),$$

normally distributed with conditional mean vector characterized by

$$\mu_{Z_1} = \psi_1 + \sum_{\tau=t}^{T-1} a_{1\tau+1}\mu_{X\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1} \left(\sum_{\iota=t}^{\tau-1} \mu_{X\iota+1} \right),$$

$$\mu_{Z_2} = \psi_2 + \sum_{\tau=t}^{T-1} \mu_{Y\tau+1},$$

and the conditional variance covariance matrix defined by

$$\sigma_{Z_1}^2 = \sum_{\tau=t}^{T-1} a_{1\tau+1}^2 \sigma_{X\tau+1}^2 + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}^2 \sum_{\iota=t}^{\tau-1} \sigma_{X\iota+1}^2$$

$$+2 \sum_{t \leq i < j \leq T-1} a_{0i+1} a_{0j+1} Cov \left(\sum_{\iota=t}^{i-1} \log \frac{C_{\iota+1}}{C_\iota}, \sum_{\iota'=t}^{j-1} \log \frac{C_{\iota'+1}}{C_{\iota'}} \right)$$

$$= \sum_{\tau=t}^{T-1} a_{1\tau+1}^2 \sigma_{X\tau+1}^2 + \sum_{\tau=t+1}^{T-1} a_{0\tau+1}^2 \sum_{\iota=t}^{\tau-1} \sigma_{X\iota+1}^2$$

$$+2 \sum_{t \leq i < j \leq T-1} a_{0i+1} a_{0j+1} \sum_{\iota=t}^{i-1} \sigma_{X\iota+1}^2, \tag{A6}$$

and

$$\sigma_{Z_2}^2 = \sum_{\tau=t}^{T-1} \sigma_{Y\tau+1}^2, \tag{A7}$$

and

$$\rho_{Z_1,Z_2} \sigma_{Z_1} \sigma_{Z_2} = Cov \left(\sum_{\tau=t}^{T-1} a_{1\tau+1} \log \left(\frac{C_{\tau+1}}{C_\tau} \right), \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau} \right)$$

$$+ Cov \left(\sum_{\tau=t+1}^{T-1} a_{0\tau+1} \sum_{\iota=t}^{\tau-1} \log \frac{C_{\iota+1}}{C_\iota}, \sum_{\tau=t}^{T-1} \log \frac{D_{\tau+1}}{D_\tau} \right)$$

$$= \sum_{\tau=t}^{T-1} a_{1\tau+1} \sigma_{XY,\tau+1} + \sum_{\tau=t+1}^{T-1} a_{0\tau+1} \left(\sum_{\iota=t}^{\tau-1} \sigma_{XY,\iota+1} \right). \tag{A8}$$

The coefficients ψ_1 and ψ_2 are defined respectively by Equations (A4) and (A5). Second, we find the distribution of (Z_2, Z_3).

Notice that

$$Z_3 = Z_1 + Z_2 + \log \frac{K}{S_t}.$$

Hence,

$$\begin{bmatrix} Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} Z_2 \\ Z_1 + Z_2 + \log \frac{K}{S_t} \end{bmatrix} = \begin{bmatrix} 0 \\ \log \frac{K}{S_t} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}. \tag{A9}$$

Since (Z_1, Z_2) is normally distributed, it follows that (Z_2, Z_3) is normally distributed with mean

$$\begin{bmatrix} 0 \\ \log \frac{K}{S_t} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_{Z_1} \\ \mu_{Z_2} \end{bmatrix},$$

and covariance matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{Z_1}^2 & \rho_{Z_1,Z_2} \sigma_{Z_1} \sigma_{Z_2} \\ \rho_{Z_1,Z_2} \sigma_{Z_1} \sigma_{Z_2} & \sigma_{Z_2}^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}'.$$

In addition,

$$\psi = Cov_t \left(\log m_t^T, \log \left(\frac{S_T + D_T}{S_T} \right) \Big| U_1^T \right) = Cov_t \left(Z_1, Z_2 | U_1^T \right)$$

$$= \rho_{Z_1,Z_2} \sigma_{Z_1} \sigma_{Z_2},$$

where $\rho_{Z_1,Z_2} \sigma_{Z_1} \sigma_{Z_2}$ is as defined in Equation (A8).

Case 2. When the pricing kernels are expressed as in Equation (30).

We first find the distribution of Z_1 and Z_2. Under Assumption 3′, the random vector,

$$
\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} =
\begin{bmatrix}
-\phi(T-t) + \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right) \\
\log\frac{\left(\varphi\left(U_1^T\right)+1\right)}{\varphi\left(U_1^{T-1}\right)} + \sum_{\tau=t}^{T-2}\log\frac{\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^{\tau}\right)} - \log\frac{K}{S_t}
\end{bmatrix}
$$

$$
+ \sum_{\tau=t}^{T-2}
\begin{bmatrix} [\rho(U_{\tau+1}) - (\rho(U_{\tau})+\alpha)] & 0 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} \log\left(\frac{C_{\tau+1}}{C_{\tau}}\right) \\ \log\left(\frac{D_{\tau+1}}{D_{\tau}}\right) \end{bmatrix}
$$

$$
+ \begin{bmatrix} -(\rho(U_{T-1})+\alpha) & 0 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} \log\left(\frac{C_T}{C_{T-1}}\right) \\ \log\left(\frac{D_T}{D_{T-1}}\right) \end{bmatrix}.
$$

is normally distributed with conditional mean and variance vectors characterized by

$$
\mu_{Z_1} = -\phi(T-t) - (\rho(U_{T-1})+\alpha)\mu_{X_T},
$$

$$
+ \sum_{\tau=t}^{T-2}[\rho(U_{\tau+1}) - (\rho(U_{\tau})+\alpha)]\mu_{X_{\tau+1}} + \rho(U_t)\log\left(\frac{C_t}{C_{t-1}}\right)
$$

$$
\mu_{Z_2} = \log\frac{\left(\varphi\left(U_1^T\right)+1\right)}{\varphi\left(U_1^{T-1}\right)} + \sum_{\tau=t}^{T-2}\log\frac{\varphi\left(U_1^{\tau+1}\right)}{\varphi\left(U_1^{\tau}\right)} + \sum_{\tau=t}^{T-1}\mu_{Y_{\tau+1}} - \log\frac{K}{S_t},
$$

and

$$
\sigma_{Z_1}^2 = (\rho(U_{T-1})+\alpha)^2\sigma_{X_T}^2 + \sum_{\tau=t}^{T-2}[\rho(U_{\tau+1}) - (\rho(U_{\tau})+\alpha)]^2\sigma_{X_{\tau+1}}^2,
$$

$$
\sigma_{Z_2}^2 = \sum_{\tau=t}^{T-1}\sigma_{Y_{\tau+1}}^2,
$$

and

$$
\rho_{Z_1,Z_2}\sigma_{Z_1}\sigma_{Z_2} = Cov\left(Z_1, Z_2 | U_1^T\right)
$$

$$
= -(\rho(U_{T-1})+\alpha)Cov\left(\log\left(\frac{C_T}{C_{T-1}}\right), \log\left(\frac{D_T}{D_{T-1}}\right)\bigg| U_1^T\right)
$$

$$
+ \sum_{\tau=t}^{T-2}[\rho(U_{\tau+1}) - (\rho(U_{\tau})+\alpha)]
$$

$$
\times Cov\left(\log\left(\frac{C_{\tau+1}}{C_{\tau}}\right), \log\left(\frac{D_{\tau+1}}{D_{\tau}}\right)\bigg| U_1^T\right)
$$

$$
= -(\rho(U_{T-1})+\alpha)\sigma_{XY,T}
$$

$$
+ \sum_{\tau=t}^{T-2}[\rho(U_{\tau+1}) - (\rho(U_{\tau})+\alpha)]\sigma_{XY,\tau+1}. \tag{A10}
$$

Since (Z_1, Z_2) is normally distributed, it follows that (Z_2, Z_3) is normally distributed [see Equation (A9)].

In addition,

$$\psi = Cov_t \left(\log m_t^T, \log \left(\frac{S_T + D_T}{S_t} \right) \Big| U_1^T \right) = Cov_t \left(Z_1, Z_2 | U_1^T \right)$$

$$= \rho_{Z_1, Z_2} \sigma_{Z_1} \sigma_{Z_2},$$

where $\rho_{Z_1, Z_2} \sigma_{Z_1} \sigma_{Z_2}$ is defined in Equation (A10). ∎

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